The Triangle: A One-Page Primer (Date: March 15, 2004[1])

An **angle** is formed by two intersecting lines. A **right** angle has 90°. An angle that is less (greater) than a right angle is called **acute** (**obtuse**). Two angles whose sum is 90° (180°) are called **complementary** (**supplementary**). Two intersecting **perpendicular** lines form right angles. The **angle bisectors** are given by the locus of points equidistant from its two lines. The **internal** and **external** angle bisectors are perpendicular to each other. The **vertically** opposite angles formed by the intersection of two lines are equal.

In triangle ABC, the **angles** are denoted A, B, C; the **sides** are denoted $a = BC$, $b = AC$, $c = AB$; the **semi-perimeter** (s) is $\frac{1}{2}$ the **perimeter** $(a + b + c)$. $A + B + C = 180°$. **Similar** figures are **equiangular** and their corresponding components are proportional. Two triangles are similar if they share two equal angles. Euclid's VI.19, the areas of similar triangles are proportional to the second power of their corresponding sides ($\frac{\Delta ABC}{\Delta A'B'C'} = \frac{a^2}{a'^2}$). An **acute** triangle has three acute angles. An **obtuse** triangle has one obtuse angle. A **scalene** triangle has all its sides unequal. An **isosceles** triangle has two equal sides. I.5 (pons asinorum): The angles at the base of an isosceles triangle are equal. An **equilateral** triangle has all its sides congruent. A **right** triangle has one right angle (at C). The **hypotenuse** of a right triangle is the side opposite the 90° angle and the **catheti** are its other legs. In a right triangle, $\sin = \frac{\text{opp}}{\text{hyp}}$, $\cos = \frac{\text{adj}}{\text{hyp}}$, $\tan = \frac{\text{opp}}{\text{adj}}$, $\cot = \frac{\text{adj}}{\text{opp}}$, $\sec = \frac{\text{hyp}}{\text{adj}}$, $\csc = \frac{\text{hyp}}{\text{opp}}$. The **median** of a triangle is a line joining a vertex to the midpoint of the opposite side. An **altitude**, h_a, h_b, or h_c, of a triangle is the line from the vertex perpendicular to the side opposite (where it meets the side is the **foot** of the altitude). The **orthic** (or **pedal**) triangle is given by the three feet of the altitudes of a given triangle. I.47 (**Pythagorean theorem**): in a right triangle, $c^2 = a^2 + b^2$. VI.6: A straight line parallel to a side of a triangle, cuts the other two sides proportionally; and conversely. **Steiner-Lehmus** theorem: any triangle with two equal angle bisectors (as measured from a vertex to its opposite side) is isosceles. Any triangle with two equal medians (or altitudes) is isosceles. Two triangles are congruent if and only if SAS (I.4) or SSS (I.8) or AAS (I.26) where A represents corresponding angles and S corresponding sides. The inscribed triangle of minimal perimeter for any acute angled triangle (**Fagnano's** problem) is the orthic triangle. The **Fermat point** of a triangle is given by the intersection of the three lines joining a vertex to the opposite vertex of the equilateral triangle outwards upon its opposite side. In a triangle where each angle is less than 120°, the Fermat point has the smallest possible sum of the distances between it and each vertex.

The Centers And Circles Of A Triangle

Construction / Description	Center		Circle	
	Notation	Name	Radius	Name
The Intersection Of				
The three medians	G	Centroid	None	None
The three altitudes	H	Orthocenter	None	None
The perpendicular bisectors of the three sides	O	Circumcenter	R	Circumcircle
The three internal angle bisectors	I	Incenter	r	Incircle
The external angle bisectors of pairs of vertices	I_a, I_b, I_c	Excenters	r_a, r_b, r_c	Excircles
The circumcircle of the orthic triangle (or the circumcircle of the midpoints of each side)	N	Nine-Point	n	Nine-Point (Feuerbach)

The **area**, Δ, of a triangle is given by $\frac{1}{2}ah_a = sr = (s-a)r_a = \frac{1}{2}bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}$ (**Heron's** Formula). $2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; $r^2 = (\frac{\Delta}{s})^2 = \frac{(s-a)(s-b)(s-c)}{s}$, ${r_a}^2 = (\frac{\Delta}{s-a})^2 = \frac{s(s-c)(s-b)}{s-a}$; $4R = r_a + r_b + r_c - r = \frac{abc}{\Delta}$; $h_a = 2R\sin B\sin C$. $OGNH$ is a line called the **Euler line** where $OG = \frac{1}{2}GH = \frac{1}{3}OH = 2GN$. The distance from G to the midpoint of a side is $\frac{1}{3}$ the median to that side. If the Euler line passes through a vertex, the triangle is either right-angled or isosceles (or both). A triangle is right angled (obtuse) if and only if $r + 2R = s$ $(r + 2R > s)$. In a right triangle, O is the midpoint of the hypotenuse and H is the right angled vertex. In an obtuse triangle, O and H are outside the triangle. The vertices of the medial and orthic triangles, and the midpoints of the lines joining H to its three vertices all lie on the nine-point circle. The orthocenter (circumcenter) of any triangle coincides with the incenter (orthocenter) of its orthic (median) triangle. The lengths from the vertex A to the points of tangency with the incircle and the three excircles are respectively $t = s - a$, $t_a = s$, $t_b = s - c$, and $t_c = s - b$. Two triangles are **perspective from a point (line)** if their three pairs of corresponding vertices (sides) are joined by (meet in) concurrent lines (collinear points). The **Simson line** of a point P on the circumcircle of a triangle joins the feet of the perpendiculars from P to each of its sides. The **Erdös-Mordell** theorem: in any triangle, ABC, with O inside and where P, Q, and R are the feet of the perpendiculars from O to BC, AC, AB, respectively, then $OA + OB + OC \geq 2(OP + OQ + OR)$. **Morley's Theorem** states the three points of intersection of the adjacent trisectors of the angles of any triangle meet to form an equilateral triangle.

[1] Compiled by Christopher J. Fearnley. The version at http://www.CJFearnley.com/triangle.primer.pdf is most current.