Discussion Questions and Problems for "Exploring Homeomorphism through Experiments on the Möbius Band"

(Revision: December 19, 2017[1])

On 23 December 2017 *Math Counts* will discuss "Exploring Homeomorphism through Experiments on the Möbius Band"[2]. The following questions and problems are based on Chapters 3-4 of Stephen Barr's fun book "Experiments in Topology". Your attempts to address them will help guide our discussion. If the book or these questions are unclear, please ask for help on the event page[2].

1. According to chapter 1 of Barr's book, what is the rule for homeomorphically distorting one surface into another? What caveats does Barr mention for using his rule? How does Barr's definition compare to Wikipedia's definition of *homeomorphism*?

2. What is the property of orientability? Which of the sphere, disk (the 2D surface bounded by a circle), cylinder, torus, and Möbius band are orientable and which are nonorientable?

3. In a paper model of a Möbius strip, what happens when you cut through the middle of the strip? How would you describe the resulting surface? What familiar surface is the result homeomorphic to?

4. Can one build a paper model of a Möbius strip whose width is greater than its length? Let us define the width of a Möbius band to be the length of either of its two opposite edges that are glued together. Let us define its length to be the length of its one and only edge. So, can the width to length ratio (width/length) ever exceed 1 (or even $\frac{1}{2}$)?

5. In the Möbius strip construction with three equilateral triangles in Figure 7 on p. 43, what is the width to length ratio?

6. In the Möbius strip construction in Figure 14 on p. 47, what is the width to length ratio? During the Nov 25th meetup, we were unable to justify the width to length result given by Barr on p. 48. Is he mistaken? What is going on with this Möbius strip?

7. What is the maximum width to length ratio possible for a Möbius strip?

8. Build a Möbius strip with the largest width to length ratio you have the patience and wherewithal to make. What is the width to length ratio in your model?

9. On p. 48, Barr suggests that the Möbius band folded from a square strip of paper has the minimal width to length ratio which can still be cut along the centerline of the strip and unfolded to a new surface (see question 3)? Why does the property of cutting a Möbius strip through its centerline fail when the width to length ratio exceeds $\frac{1}{2}$?

10. In the text on p. 49 and the associated Figures 15 and 16, a puzzle about cutting a Möbius strip into two equal area pieces is described. What is the solution to the puzzle? What does it mean for the cut to start on the edge? What logic explains the puzzle? What did you learn from this puzzle?

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[1]Note: Please let me know of any difficulties. There may be a revised version correcting issues if any are found. Compiled by CJ Fearnley. `http://blog.CJFearnley.com`.

[2]`http://www.meetup.com/MathCounts/events/243186341`

11. In considering the variations on a Möbius band in chapter 3, how should we think about the connectedness of the joined (glued) edge of a Möbius band? How is the homeomorphic property preserved in each of these variations? Are you sure? Why?

12. In considering the sequence of experiments in chapter 3 where Möbius strips of decreasing length to width ratios (or increasing width to length ratios) are considered, what did you learn about the nature of homeomorphism and paper representations of topological surfaces?

13. Why are the paper models of the Klein Bottle and Projective Plane in Figures 2 and 3 on p. 51 considered troublesome? Is the difficulty related to the subtleties of connectedness in the definition of a homeomorphism? How would you explain the problem?

14. What does Barr's text mean on p. 52 where it says "some meaningful restrictions must be placed on the Moebius strip, too, as to how much of the edges ought to be joined" and in wondering "if the amount of edge involved can be *increased*"? What does it mean to increase or decrease the amount of edge involved? What are the restrictions alluded to?

15. How can a Möbius strip be constructed from an annulus with a radial cut through its bounding circles as in Figure 4 on p. 52? In what way does this experiment test the question about the amount of edge involved in forming a Möbius band?

16. How can a Möbius strip be constructed from a disk with a radial cut to its center? What is the width to length ratio in the resulting Möbius band?

17. How can a Möbius band be constructed from a semicircle as in Figure 11 on p. 55? What is the width to length ratio in the resulting Möbius strip?

18. How can a Möbius strip be constructed from a 30° sector of a circle? What is the width to length ratio in the resulting Möbius band?

19. On p. 61, Barr concludes chapter 4 by saying "The moral of all this is that when we allow *only one kind* of distortion (bending), unexpected relationships persist. Suspicion arises that with *any* distortion allowed, what persists must be invariant indeed, *and perhaps overlooked before*." How do you interpret this conclusion? What invariants did you infer from the experiments into topology that you undertook in reading chapters 3 and 4?

20. In considering the variations on a Möbius band in chapter 4 and any additional thought or model-building experiments you might have undertaken, how should we think about the distortion and connectedness in the joined edge of the strip of paper used to make a Möbius band? How is the homeomorphic property preserved in each of these variations?

21. In considering the sequence of experiments discussed in chapter 4 where Möbius strips with various extents of connectedness are considered, what did you learn about the nature of homeomorphism, topological invariants, and paper representations of topological surfaces?

22. Given the considerations in chapters 3 and 4, what subtleties, limitations, and caveats must we heed about the nature of the distortions allowed and the requirements for connectedness and continuity in Stephen Barr's definition of a homeomorphism?

23. Inspired by the book or these questions or your own initiative, what additional experiments did you attempt? What did you learn from these supplemental experiments?